TORSION OF THE KHOVANOV HOMOLOGY

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Abstract. Khovanov homology is a recently introduced invariant of oriented links in S^3. Its (graded) Euler characteristic is a version of the Jones polynomial of the link. In this paper we study torsion of the Khovanov homology. Based on our calculations, we conjecture that

- every link except the trivial knot, the Hopf link, their connected sums and disjoint unions has torsion of order 2;
- no link has torsion of odd order;
- all homologically thin links are torsion thin, in particular, their torsion is completely determined by the free part of the Khovanov homology;
- a knot is torsion rich if and only if its reduced Khovanov homology has torsion;
- two knots have the same ranks of the Khovanov homology if and only if they have the same torsion.

We prove the first two conjectures for all non-split alternating links. We also prove a weakened version of the third conjecture, namely, that all homology slim links are weakly torsion thin.

1. Introduction

Let L be an oriented link in the Euclidean space represented by a diagram D. In [6] Mikhail Khovanov assigned to D a family of Abelian groups $\mathcal{H}^{i,j}(D)$, whose isomorphism classes depend on the isotopy class of L only. These groups are defined as homology groups of an appropriate (graded) chain complex $\mathcal{C}(D)$ with $\mathbb{Z}$ coefficients. Groups $\mathcal{H}^{i,j}(D)$ are nontrivial for finitely many values of the pair (i,j) only. The graded Euler characteristic of $\mathcal{C}(D)$ is a version of the Jones polynomial of L:

$$K_L(q)=\sum_{i,j}(-1)^i q^j h^{i,j}(L), \tag{1.1}$$

where $h^{i,j}(L)$ are the Betti numbers of $\mathcal{H}^{i,j}(L)$, i.e. $h^{i,j}(L)=\dim_{\mathbb{Q}}(\mathcal{H}^{i,j}(L)\otimes\mathbb{Q})$. The (Laurent) polynomial $K_L(q)$ is completely determined by the following identities:

$$-q^{-2}K_{\times_+}(q)+q^2K_{\times_-}(q)=(q-1/q)K_{)(_0}(q);\qquad K_{\bigcirc}(q)=q+1/q. \tag{1.2}$$

Please refer to [2, 15] for more information.

Already the first examples computed showed that the Khovanov homology possesses some unexpected and fascinating properties. In particular, several numerological conjectures were formulated. We recall them briefly below.

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For a given link L the ranks $h^{i,j}(L)$ of the Khovanov homology can be arranged into a table with columns and rows numbered by i and j, respectively (see, for example, Figure 1). A pair of entries in such a table is said to be a *"knight move" pair*, if these entries have the same value $a > 0$ and their i- and j-positions in the table differ by 1 and 4, respectively. This "knight move" rule is depicted in Figure 1.

	a
a	

$\mathbf{7_7}$	-3	-2	-1	0	1	2	3	4
9								1
7							1	
5						2	1	
3					2	1		
1				1+1	2			
-1			2	1+2				
-3		1	1					
-5		2						
-7	1							

Figure 1. Pattern of the "knight move" rule. Ranks of the Khovanov homology of the knot 7_7 that illustrates Conjecture 1.A.

1.A. Conjecture (Bar-Natan, Garoufalidis, Khovanov [2]). *Let L be a knot. Consider the table of Khovanov ranks $h^{i,j}(L)$ of L. If one subtracts 1 from two adjacent entries in the column $i = 0$, then the remaining entries are arranged in* "knight move" pairs *(see, for example, Figure 1 where the 1's to subtract are shown on a gray background and the rest of circles joined by lines depict the "knight move" pairs).*

Remark. In fact, different "knight move" pairs are allowed to overlap. In this case the common entry is simply the sum of the overlapping entries from both pairs. For example, the knot $13n_{3663}$ [1], whose homology are presented in section A.4 from the Appendix, has two overlapping pairs $(h^{1,-1}, h^{2,3})$ and $(h^{2,3}, h^{3,7})$. This confusion will be cleared up after we give a more rigorous statement of this Conjecture in section 2.3.

Remark. Conjecture 1.A was proved by Eun Soo Lee [9] for the special case of H-thin knots (see below), in particular for all alternating knots.

1.B. Definition (Khovanov [7]). A link L is said to be *homologically thin* or simply *H-thin* if its nontrivial homology groups lie on two adjacent diagonals. A *b-diagonal* or simply a *diagonal* is the line defined by $2i - j = b$ for some b. A link L that is not H-thin is said to be *H-thick*.

[1] Throughout this paper we use the following notation for knots: knots with 10 crossings or less are numbered according to the Rolfsen's knot table [10] and knots with 11 crossings or more are numbered according to the knot table from Knotscape [4]. For example, the knot 9_{42} is the knot number 42 with 9 crossings from the Rolfsen's table and the knot $13n_{3663}$ is a non-alternating knot number 3663 with 13 crossings from the Knotscape's one.

The knot 7_7 is an example of an H-thin knot, since its homology is supported on the (±1)-diagonals (see Figure 1). The first H-thick knot is 8_{19} (see Figure 2).

1.C. DEFINITIONS. Let F be a ring or a field. A link L is said to be *homologically thin with coefficients in* F or simply F*H-thin* if its nontrivial homology groups with coefficients in F lie on two adjacent diagonals. A link L is said to be *homologically slim* or simply *H-slim* if it is H-thin and all its homology groups on the upper diagonal have no torsion.

1.D. *If a link is H-thin (this is the same as being $\mathbb{Z}$H-thin) then it is $\mathbb{Q}$H-thin as well. If a link is H-slim then it is $\mathbb{Z}_p$H-thin for every prime p.*

All known examples of H-thin knots are H-slim. It is not clear why this should be the case in general.

1.E. THEOREM (Lee [8]). *Every oriented non-split alternating link L is H-slim and the Khovanov homology of L are supported on $(\sigma(L)\pm1)$-diagonals, where $\sigma(L)$ is the signature of L.*

Remark. This theorem was originally conjectured by Bar-Natan, Garoufalidis, and Khovanov [2, 3] in a somewhat weaker formulation. Their conjecture stated that every non-split alternating link is H-thin and not H-slim. Lee proved a stronger version (see [8], Corollary 4.3).

Most of the results and observations obtained about the Khovanov homology were about its free part. Torsion of the Khovanov homology did not enjoy the same attention mostly because it is more difficult to compute. Nonetheless, using the methods of [13] and the program `KhoHo` [12] based on these methods, the author was able to compute the torsion for all prime links with up to 11 crossings, all prime knots with up to 13 crossings and also many thousands of other knots with up to 19 crossings.

Initial results show that the torsion is at least as interesting and important as the ranks of the Khovanov homology. First of all, every knot and link considered, except the unknot, the Hopf link, their connected sums, and disjoint unions, has torsion and, moreover, torsion elements of order 2 are always present. If proved, this could lead to an easy way to detect the unknot. There is only one other torsion order that appeared so far: 16 knots with 15 crossings and 2 knots with 16 crossings are known to have torsion of order 4. All of them are braid positive knots. Khovanov homology of one of the first knot with torsion of order 4, the $(4,5)$-torus knot, is presented in section A.5 from the Appendix. It is natural to assume that only powers of 2 can be orders of Khovanov homology classes.

We combine these observations into the following conjectures.

Conjecture 1. *Khovanov homology of every non-split link except the trivial knot, the Hopf link, and their connected sums has 2-torsion, that is, torsion elements of order 2.*

Conjecture 2. *Khovanov homology of every link has no torsion of order p for any p other than a power of 2.*

Let $t_p^{i,j}(L)$, where p is a power of a prime number, be the p-rank of $\mathcal{H}^{i,j}(L)$, that is, the multiplicity of $\mathbb{Z}_p$ in the canonical decomposition of $\mathcal{H}^{i,j}(L)$. Let also $T_p^{i,j}(L)=\sum_{k=1}^{\infty} t_{p^k}^{i,j}(L)$ for a prime p. The complete information about the canonical decomposition of all the groups $\mathcal{H}^{i,j}(L)$ can be combined into a table. The

	a
	[a]
a	

$\mathbf{8_{19}}$	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				[1]	1	
9			1			
7	1					
5	1					

$\mathbf{9_{42}}$	-4	-3	-2	-1	0	1	2
7							1
5							[1]
3					1	1	
1				1	1 [1]		
-1				1 [1]	1		
-3		1	1				
-5		[1]					
-7	1						

FIGURE 2. Torsion version of the "knight move" rule. Khovanov homology of the knots 8_{19} and 9_{42} that are both H-thick but are T-thick and T-thin, respectively.

columns and rows of such a table are numbered by i and j, respectively. The table entries contain the corresponding rank $h^{i,j}(L)$ and a comma separated list of $t_p^{i,j}(L)$ for all relevant p, put into square brackets (see, for example, Figure 2). Empty list, that is, no square brackets means that the corresponding group has no torsion. Most of the knots and links have 2-torsion only and, hence, such torsion lists contain at most one entry for them.

Similar to the case of ranks, most of the knots and links have very simple torsion. By analogy we call such knots and links *T-thin.*

1.F. DEFINITION. A link L is said to be *weakly torsion thin* or simply *WT-thin* if it satisfies Conjecture 1.A, has no torsion of odd order, and for every "knight move" pair of value a that comprises entries (i,j) and $(i{+}1,j{+}4)$, one has $T_2^{i+1,j+2}(L)=a$, such that all 2^k-ranks of L are obtained in this way (see Figure 2, where the torsion corresponding to a "knight move" pair is depicted in a gray square). L is said to be *torsion thin* or *T-thin* if it is WT-thin and has torsion of order 2 only. It follows that in this case $t_2^{i+1,j+2}(L)=a$. A link L that is not WT-thin is said to be *T-thick.*

Remark. Strictly speaking, Conjecture 1.A is stated for knots only. Nonetheless, it was generalized (and proved) by Lee [9] to the case of H-thin links (see Theorem 2.3.C). Throughout this paper we are going to refer to a link L as being T-thin or T-thick with understanding that this notion is assumed to be applicable, that is, L is either a knot or an H-thin link.

The knot 9_{42} is T-thin (see Figure 2) and the knot 8_{19} is the first T-thick one. Both of them are H-thick. There are 6 T-thick knots with 10 crossings, 10 with 11 crossings, 71 with 12 crossings, and 322 with 13 crossings. All of them are H-thick as well. The torsion of a T-thin link is completely determined by the ranks of the Khovanov homology. In particular, for non-split alternating links it is completely determined by the Jones polynomial and the signature.

Conjecture 3. *Every H-thin link is T-thin. In particular, every non-split alternating link is T-thin.*

Most of the T-thick links have their torsion ranks never greater than the value of the corresponding "knight move" pair.

1.G. Definition. A link L is said to be *torsion rich* or simply *T-rich* if it satisfies Conjecture 1.A, has no torsion of odd order (that is, all torsion elements have order 2^k for some k) and there is at least one value of (i,j) such that $T_2^{i+1,j+2}(L)$ is *greater* that the value of the corresponding "knight move" pair $(h^{i,j}, h^{i+1,j+4})$.

The first T-rich knot has 13 crossings. It is the knot $13n_{3663}$ mentioned above. This knot is also the first one whose homology occupy 4 diagonals.

In [7] Khovanov defined a reduced version of his homology. The graded Euler characteristic of this reduced homology is again a version of the Jones polynomial. In particular, it satisfies the skein relation from (1.2) but is normalized to be 1, as opposite to $q+1/q$, on the unknot. The reduced Khovanov homology always occupy exactly one diagonal less than the standard ones. Very few knots have torsion in the reduced homology and all known examples of those that have are T-rich. There are no known examples of links that have torsion of order other than 2 in their reduced homology.

Conjecture 4. *A knot is T-rich iff its reduced Khovanov homology has torsion.*

The following conjecture is a bit optimistic, but its confirmation would be very exciting, since the torsion seems to be easier to read from the diagram than the rest of the homology.

Conjecture 5. *Two prime knots have the same ranks of the Khovanov homology iff they have the same torsion.*

Among about 12'000 of prime knots for which the Khovanov homology was already calculated, there are almost 7'000 pairs (not counting mirror images) having the same ranks. They are exactly the pairs of knots that have the same torsion.

In this paper we prove the following results.

Theorem 1 (cf. Conjecture 2)**.** *Khovanov homology of every H-slim link has no torsion or order p for any p other than a power of* 2.

Corollary 2. *Khovanov homology of every non-split alternating link has no torsion or order p for any p other than a power of* 2.

We prove Theorem 1 by showing that for every H-slim link Conjecture 1.A holds true not only for homology over $\mathbb{Z}$, but also for homology over $\mathbb{Z}_p$ for all odd prime p. This can be proved using a slight modification of Lee's methods from [9].

Theorem 3 (cf. Conjecture 3)**.** *Every H-slim link is WT-thin.*

Corollary 4. *Every non-split alternating link is WT-thin.*

Corollary 5 (cf. Conjecture 1)**.** *Every alternating link except the trivial knot, the Hopf link, their connected sums and disjoint unions has torsion of order* 2.

Remark. Marta Asaeda and Jozef Przytycki gave recently [1] an independent proof of Corollary 5. Moreover, they proved that an adequate link that satisfies some additional conditions has torsion of order 2 as well. Contrary to our approach, their proof is constructive. They explicitly find a generator in the Khovanov chain complex that gives rise to an appropriate torsion element.

Theorem 3 is a corollary of Theorem 3.2.A that establishes a structure of an exact sequence on Khovanov homology over $\mathbb{Z}_2$. It remains only to show that every

H-thin link is H-slim and that no H-thin link has torsion elements of order 2^k for $k \geq 2$ to prove Conjecture 3 completely.

This paper is organized as follows. Section 2 contains main definitions and facts about the Khovanov homology that are going to be used in the paper. Theorems 1 and 3 and Corollary 5 are proved in sections 4, 3, and 3.3, respectively. The Appendix contains information about standard and reduced Khovanov homology of knots whose torsion has some remarkable properties.

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2. Khovanov chain complex and its properties

In this section we briefly recall the main ingredients of the Khovanov homology theory. Our exposition follows the one by Viro, whose paper [15] is recommended for a full treatment.

2.1. Generators and the differential of the Khovanov chain complex. Let D be a diagram representing an oriented link L. Assign a number ± 1, called a *sign*, to every crossing of D according to the rule depicted in Figure 3. The sum of such signs over all the crossings is called the *writhe number* of D and is denoted by $w(D)$.

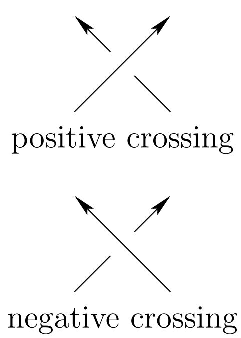


Figure 3. Positive and negative crossings

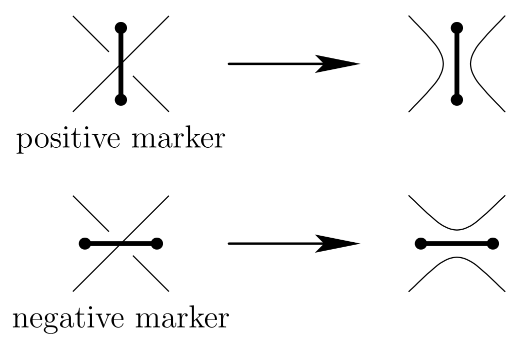


Figure 4. Positive and negative markers and the corresponding smoothings of a diagram.

At every crossing of D the diagram locally divides the plane into 4 quadrants. A choice of a pair of antipodal quadrants is shown on the diagram with the help of *markers*, which can be either *positive* or *negative* (see Figure 4). A collection of markers chosen at every crossing of a diagram D is called a *(Kauffman) state* of D. There are, obviously, 2^n different states, where n is the number of crossings of D. Denote by $\sigma(s)$ the difference between the numbers of positive and negative markers in a given state s.

Given a state s of a diagram D, one can smooth D at every crossing with respect to the corresponding marker from s (see Figure 4). The result is a family D_s of disjointly embedded circles. Denote the number of these circles by $|s|$.

Let s be a state of a diagram D. Equip each circle from D_s with either a plus or minus sign. We call the result an *enhanced (Kauffman) state* of D that belongs

to s. There are exactly $2^{|s|}$ different enhanced states that belong to a given state s. Denote by $\tau(S)$ the difference between the numbers of positively and negatively signed circles in a given enhanced state S.

With every enhanced state S belonging to a state s of a diagram D one can associate two numerical characteristics:

$$i(S) = \frac{w(D) - \sigma(s)}{2}, \qquad j(S) = -\frac{\sigma(s) + 2\tau(S) - 3w(D)}{2}.$$

Since both $w(D)$ and $\sigma(s)$ are congruent modulo 2 to the number of crossings, the numbers $i(S)$ and $j(S)$ are always integer.

Fix $i, j \in \mathbb{Z}$. It was shown by Viro [15] that the Khovanov chain group $\mathcal{C}^{i,j}(D)$ is generated by all the enhanced states of D with $i(S) = i$ and $j(S) = j$. With the basis of the chain groups chosen, the Khovanov differential $d^{i,j} : \mathcal{C}^{i,j}(D) \to \mathcal{C}^{i+1,j}(D)$ can be described by its matrix, called the *incidence matrix* in this context. The elements of the incidence matrix are called *incidence numbers* and are denoted by $(S_1 : S_2)$, where S_1 and S_2 are enhanced states (that is, generators) from $\mathcal{C}^{i,j}(D)$ and $\mathcal{C}^{i+1,j}(D)$, respectively.

The incidence number $(S_1 : S_2)$ is zero unless all of the following 3 conditions are met:

I. the markers from S_1 and S_2 differ at one crossing of D only and at this crossing the marker from S_1 is positive, while the marker from S_2 is negative;

Remark. If this condition is fulfilled, D_{S_2} is obtained from D_{S_1} by either joining two circles into one or splitting one circle into two and, hence, $|S_2| = |S_1| \pm 1$.

II. the common circles of D_{S_1} and D_{S_2} have the same signs;

III. one of the following 4 conditions are met:
   1) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} are negative and the resulting circle from D_{S_2} is negative as well;
   2) $|S_2| = |S_1| - 1$, the joining circles from D_{S_1} have different signs and the resulting circle from D_{S_2} is positive;
   3) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} is positive and both the resulting circles from D_{S_2} are positive as well;
   4) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} is negative and the resulting circles from D_{S_2} have different signs.

If all three conditions above are fulfilled, the incidence number $(S_1 : S_2)$ is equal to $(-1)^t$, where t is defined as follows. Choose some order on the crossings of D. Let the crossing, where one changes the marker to get from S_1 to S_2, have number k in this order. Then t is the number of negative markers in S_1 whose order number is greater than k. As it turns out, the resulting homology does not depend on the choice of the crossing order. More details can be found in [2, 15].

2.2. Reduced Khovanov homology. Let D be a diagram of a link L. Pick a base point on D that is not a crossing. Let $\widetilde{\mathcal{C}}(D)$ be a subcomplex of $\mathcal{C}(D)$ generated by all the enhanced states of D that have a positive sign on the circle that the base point belongs to. The homology $\widetilde{\mathcal{H}}(L)$ of this subcomplex is called the *reduced Khovanov homology* of L. It can be shown that if L is a knot, then its reduced homology does not depend on the choice of the base point. In general, it depends on the component of the link, where the base point is chosen. Please refer to [7] for more details.

2.2.A. (Khovanov [7], cf. (1.1)). *The graded Euler characteristic of $\widetilde{\mathcal{C}}(D)$ is the Jones polynomial of L:*

$$J_L(q) = K_L(q)/(q+1/q) = \sum_{i,j} (-1)^i q^j \widetilde{h}^{i,j}(L). \tag{2.1}$$

where $\widetilde{h}^{i,j}(L)$ are the Betti numbers of $\widetilde{\mathcal{H}}^{i,j}(L)$.

$J_L(q)$ is completely determined by the following identities (cf. (1.2)):

$$-q^{-2}J_{+}(q) + q^2 J_{-}(q) = (q-1/q)J_{0}(q); \qquad J_{\bigcirc}(q) = 1. \tag{2.2}$$

2.2.B. (Khovanov [7]). *For any link L its reduced Khovanov homology $\widetilde{\mathcal{H}}(L)$ occupies exactly one diagonal less than the standard one.*

2.2.C. Corollary. *If L is an H-thin link (in particular, a non-split alternating link), then $\widetilde{\mathcal{H}}(L)$ is supported on exactly one diagonal. If follows that $J_L(q)$ is alternating, that is, its coefficients have alternating signs. More precisely, $J_L(q) = \sum_{i\in\mathbb{Z}} c_i q^{2i+\gamma}$, where γ is the number of components of L modulo 2. Then $J_L(q)$ is alternating iff $(-1)^{i-j}c_ic_j \geq 0$ for all i and j.*

2.3. Khovanov polynomial and its torsion version. Let L be a link and $Kh(L)(t,q) = \sum_{i,j} t^i q^j h^{i,j}(L)$ be the Poincaré polynomial in variables t and q of its Khovanov homology. This polynomial is called the *Khovanov polynomial* of L. Now Conjecture 1.A can be reformulated in the following way.

2.3.A. (Rigorous statement of Conjecture 1.A). *Let L be a knot. Then there exists a polynomial $Kh'(L)$ in $t^{\pm1}$ and $q^{\pm1}$ with non-negative coefficients only and an even integer $s = s(L)$ such that*

$$Kh(L) = q^{s-1}\big(1 + q^2 + (1+tq^4)Kh'(L)\big). \tag{2.3}$$

In other words, there exist non-negative integers $g^{i,j}(L)$ such that

$$h^{i,j}(L) = g^{i,j}(L) + g^{i-1,j-4}(L) + \varepsilon^{i,j}, \tag{2.4}$$

where $\varepsilon^{0,s\pm1} = 1$ and $\varepsilon^{i,j} = 0$ if $i \neq 0$ or $j \neq s\pm 1$.

Remark. It is clear from the construction that $g^{i,j}(L)$ is the coefficient of the term $t^i q^{j-s+1}$ in $Kh'(L)$. It must be non-zero for finitely many values of the pair (i,j) only.

2.3.B. *If L is H-thin then the polynomial $Kh'(L)$ contains powers of tq^2 only. Let $Kh'(L) = \sum_{i\in Z} a_i t^i q^{2i}$. In this case $g^{i,2i+s-1}(L) = a_i$ and, hence,*

$$h^{i,2i+s-1}(L) = g^{i,2i+s-1}(L) + g^{i-1,2i+s-5}(L) + \varepsilon^{i,2i+s-1} = a_i + \delta^i; \tag{2.5}$$

$$h^{i,2i+s+1}(L) = g^{i,2i+s+1}(L) + g^{i-1,2i+s-3}(L) + \varepsilon^{i,2i+s+1} = a_{i-1} + \delta^i, \tag{2.6}$$

where $\delta^0 = 1$ and $\delta^i = 0$ if $i \neq 0$. All other $g^{i,j}(L)$ and $h^{i,j}(L)$ are zero.

Theorem 1.E implies that $s(L) = -\sigma(L)$ for all alternating knots L, where $\sigma(L)$ is the signature of L.

The following theorem is a counterpart of Conjecture 1.A for the case of H-thin links.

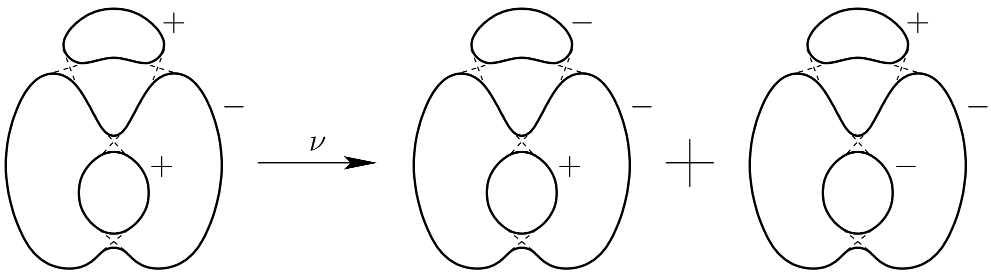


FIGURE 5. Action of ν on generators of the chain groups.

2.3.C. THEOREM (Lee [9]). *Let L be an m-component oriented H-thin link, for example, a non-split alternating link. Let $\ell_{k,l}$ be the linking number of the k-th and l-th components of L. Then*

$$Kh(L) = q^{-\sigma(L)-1}\left[(1+q^2)\left(\sum_{E\subset\{2,\dots,m\}} (tq^2)^{2\sum_{\substack{k\in E\\ l\notin E}}\ell_{k,l}}\right) + (1+tq^4)Kh'(L)(tq^2)\right] \tag{2.7}$$

for some polynomial $Kh'(L)$ with non-negative coefficients. Here $\sigma(L)$ is the signature of L.

2.3.D. DEFINITION. For a given link L, its *torsion Khovanov polynomial Kh_T* in variables $t^{\pm1}$ and $Q_p^{\pm1}$ is defined as $Kh_T(L)(t, Q_2, Q_3, \dots) = \sum_{i,j,p} t^i Q_p^j t_p^{i,j}(L)$, where i and j are arbitrary and p goes through all the powers of prime numbers. Recall that $t_p^{i,j}(L)$ is the p-rank of $\mathcal{H}^{i,j}(L)$ and $T_p^{i,j}(L) = \sum_{k=1}^{\infty} t_{p^k}^{i,j}(L)$ if p is prime.

2.3.E. 1. *A link L is T-thin iff $Kh_T(L)$ depends on the variables $t^{\pm1}$ and $Q_2^{\pm1}$ only and $Kh_T(L)(T, q) = tq^{s+1}Kh'(L)$. These polynomials are equal iff $t_p^{i,j}(L) = g^{i-1,j-2}(L)$ for all i and j (see 2.3.B).*
2. *A link L is WT-thin iff $Kh_T(L)$ depends on the variables $t^{\pm1}$ and $Q_{2^k}^{\pm1}$ only and $Kh_T(L)(t, q, q, \dots) = tq^{s+1}Kh'(L)$. These polynomials are equal iff $T_p^{i,j}(L) = g^{i-1,j-2}(L)$ for all i and j.*
3. *A link L is T-rich iff $Kh_T(L)$ depends on the variables $t^{\pm1}$ and $Q_{2^k}^{\pm1}$ only and $tq^{s+1}Kh'(L) - Kh_T(L)(t, q, q, \dots)$ contains some terms with negative coefficients.*

3. KHOVANOV HOMOLOGY WITH $\mathbb{Z}_2$ COEFFICIENTS

Denote by $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L)$ the Khovanov homology over $\mathbb{Z}_2$ (instead of $\mathbb{Z}$) of an oriented link L and by $h^{i,j}_{\mathbb{Z}_2}(L)$ the corresponding Betti numbers. In this section we construct an acyclic differential $\overline{\nu}$ of bidegree $(0, 2)$ on $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L)$ and prove Theorem 3.

3.1. Construction of the $(0, 2)$-differential. Let D be a diagram of a link L and S be some enhanced state of D. Denote by $\mathcal{N}(S)$ the set of all enhanced states that have the same markers and signs on all the circles as S does except one circle where '+' is replaced with '−'. It is obvious, that for every enhanced state $S' \in \mathcal{N}(S)$ one has $\sigma(S') = \sigma(S)$ and $\tau(S') = \tau(S) - 2$. Hence, $i(S') = i(S)$ and $j(S') = j(S) + 2$.

3.1.A. Definition. A bidegree $(0,2)$ differential $\nu^{i,j}:\mathcal{C}^{i,j}(D;\mathbb{Z}_2)\to\mathcal{C}^{i,j+2}(D;\mathbb{Z}_2)$ on the Khovanov chain groups with coefficients in $\mathbb{Z}_2$ is defined on the generators of $\mathcal{C}^{i,j}(D;\mathbb{Z}_2)$ as $\nu^{i,j}(S)=\sum_{S'\in\mathcal{N}(S)}S'$ (see Figure 5).

It is obvious that ν is indeed a differential, i.e. $\nu^2=0$. To see this we observe that for every enhanced states S and S'' that have the same markers and signs on all the circles except two circles where S has '+', while S'' has '−', S'' appears exactly twice in $\nu(\nu(S))$.

3.1.B. Lemma. *ν is acyclic, i.e. all homology groups with respect to ν are trivial.*

Proof. Let $\mathcal{Z}_n$ be the set of all length n sequences of signs '+' and '−' and let $\mathcal{Z}_n^k\subset\mathcal{Z}_n$ consists of all sequences with the difference between the numbers of minuses and pluses being exactly k (with $k\equiv n \mod 2$). $\mathcal{Z}_n$ and $\mathcal{Z}_n^k$ have 2^n and $b_n^k=\begin{pmatrix} n\\ \frac{n+k}{2}\end{pmatrix}$ elements, respectively. Denote by G_n^k the group $\mathbb{Z}_2^{b_n^k}$ whose factors are enumerated by the elements from $\mathcal{Z}_n^k$.

One can construct a differential $\mu_n^k:\mathcal{Z}_n^k\to\mathcal{Z}_n^{k+2}$ similarly to ν: a generator of G_n^k corresponding to a sequence p from $\mathcal{Z}_n^k$ is mapped into the sum of the generators of G_n^{k+2} corresponding to all the sequences obtained from p by changing exactly one '+' into '−'.

Let $\mathcal{G}_n$ be the complex

$$0\to G_n^{-n}\xrightarrow{\mu_n^{-n}}G_n^{-n+2}\xrightarrow{\mu_n^{-n+2}}\cdots\xrightarrow{\mu_n^{n-4}}G_n^{n-2}\xrightarrow{\mu_n^{n-2}}G_n^n\to 0 \tag{3.1}$$

It is clear that the complex

$$\cdots\xrightarrow{\nu^{i,j-4}}\mathcal{C}^{i,j-2}(D,\mathbb{Z}_2)\xrightarrow{\nu^{i,j-2}}\mathcal{C}^{i,j}(D,\mathbb{Z}_2)\xrightarrow{\nu^{i,j}}\mathcal{C}^{i,j+2}(D,\mathbb{Z}_2)\xrightarrow{\nu^{i,j+2}}\cdots \tag{3.2}$$

is isomorphic to a direct sum of $\mathcal{G}_{|s|}$ with various shifts, where s runs over all the Kauffman states of D such that $i(s)=i$.

We claim that the complex $\mathcal{G}_n$ is acyclic. Indeed, it is obvious for $n=1$. Let us prove this for a general n by induction. Denote by $\mathcal{G}_n^-$ the subcomplex of $\mathcal{G}_n$ that is obtained by choosing only those sequences that have '−' it the first position. Then $\mathcal{G}_n^-$ is isomorphic to $\mathcal{G}_{n-1}$ and, hence, is acyclic by the induction hypothesis. It follows that $\mathcal{G}_n$ has the same homology as $\mathcal{G}_n/\mathcal{G}_n^-$. But the later is again isomorphic to $\mathcal{G}_{n-1}$. Hence, $\mathcal{G}_n$ is acyclic as well.

This completes the proof of the lemma. □

3.1.C. Lemma. *ν commutes with the Khovanov differential d (over $\mathbb{Z}_2$).*

The proof is elementary and is left to the reader as an exercise.

3.2. Patterns in $\mathbb{Z}_2$ homology. Lemma 3.1.C implies that ν can be extended to the $(0,2)$-differential $\overline{\nu}$ on the Khovanov homology over $\mathbb{Z}_2$. This differential is also acyclic, although this does not follow from Lemma 3.1.B directly.

3.2.A. Theorem. *$\overline{\nu}$ is acyclic. In particular, for every fixed i the following sequence is exact:*

$$\cdots\xrightarrow{\overline{\nu}^{i,j-4}}\mathcal{H}_{\mathbb{Z}_2}^{i,j-2}(L)\xrightarrow{\overline{\nu}^{i,j-2}}\mathcal{H}_{\mathbb{Z}_2}^{i,j}(L)\xrightarrow{\overline{\nu}^{i,j}}\mathcal{H}_{\mathbb{Z}_2}^{i,j+2}(L)\xrightarrow{\overline{\nu}^{i,j+2}}\cdots \tag{3.3}$$

Consequently, $\sum_{j\in\mathbb{Z}}(-1)^j h_{\mathbb{Z}_2}^{i,2j+\gamma}(L)=0$ for every i, where γ is the number of components of L modulo 2.

The following proof is due to Khovanov. It replaced the original one that was too technical and unnecessary complicated.

PROOF. Choose a base point b somewhere on the diagram D away from the crossings. In [7] Khovanov introduced another differential $X^{i,j} : \mathcal{C}^{i,j}(D) \to \mathcal{C}^{i,j-2}(D)$ of bidegree $(0, -2)$ on the chain groups $\mathcal{C}^{i,j}(D)$ with $\mathbb{Z}$ coefficients. It is defined as follows. Let $S \in \mathcal{C}^{i,j}(D)$ be some enhanced state. If the circle of S that contains b has a positive sign, then $X^{i,j}(S) = 0$. Otherwise $X^{i,j}(S) = S' \in \mathcal{C}^{i,j-2}(D)$, where S' is obtained from S by changing the sign of the circle that contains b from '$-$' to '$+$'. It is obvious from the definition that $X \circ X = 0$, that is, X is indeed a differential.

It is easy to check that X commutes with the Khovanov differential d and, hence, can be extended to the $(0, -2)$-differential $\overline{X}$ on the Khovanov homology.

We claim that $\nu \circ X + X \circ \nu = \mathrm{id}$ modulo 2. Indeed, let S be some enhanced state of D. If the circle of S that contains b has a positive sign, then $\nu(X(S)) = 0$. Moreover, $\nu(S)$ is a sum of enhanced states that all but one have positive signs on their circles that contain b. Hence, $X(\nu(S)) = S$. On the other hand, if the circle of S containing b has a negative sign, then all the enhanced states from $\nu(S)$ have negative signs on their circles containing b and $X(\nu(S))$ is the sum of all the states that are obtained from S by changing the sign of the circle that contains b from '$-$' to '$+$' and changing the sign of some other circle from '$+$' to '$-$'. Moreover, $X(S)$ has one more negative sign than S and $\nu(X(S))$ is the sum of all the same states as $X(\nu(S))$ plus S itself. The claim follows.

Since ν and X both commute with the differential d, one has that $\overline{\nu}\circ\overline{X}+\overline{X}\circ\overline{\nu} = \mathrm{id}$ as well. It follows that $\overline{\nu}$ is acyclic. Indeed, if $\alpha \in \mathcal{H}^{i,j}(D)$ such that $\overline{\nu}(\alpha) = 0$, then $\overline{\nu}(\overline{X}(\alpha)) = \alpha$, that is, α lies in the image of $\overline{\nu}$. □

Remark. $\overline{X}$ mod 2 is acyclic as well, of course.

3.2.B. COROLLARY. *Let L be an H-slim link. Then it is $\mathbb{Z}_2$H-thin by 1.D, that is, its $\mathbb{Z}_2$-homology are supported on the $(-s \pm 1)$-diagonals. Theorem 3.2.A implies that $h^{i,2i+s-1}_{\mathbb{Z}_2}(L) = h^{i,2i+s+1}_{\mathbb{Z}_2}(L)$ for every i.*

PROOF OF THEOREM 3. Let L be an H-slim knot. It follows from Theorem 1 that L has torsion of order 2^k only. It remains to show that $T^{i,j}_2(L) = g^{i-1,j-2}(L)$ for all i and j (see 2.3.E). Since $T^{i,j}_2(L) = g^{i-1,j-2}(L) = 0$ for $j \neq 2i + s - 1$ and $g^{i-1,2i+s-3}(L) = a_{i-1}$ in notation of 2.3.B, we only need to prove that $T^{i,2i+s-1}_2(L) = a_{i-1}$.

Observe now that $h^{i,j}_{\mathbb{Z}_2}(L) = h^{i,j}(L) + T^{i,j}_2(L) + T^{i+1,j}_2(L)$. It follows from 2.3.B that $h^{i,2i+s-1}_{\mathbb{Z}_2} = a_i + \delta^i + T^{i,2i+s-1}_2$ and $h^{i,2i+s+1}_{\mathbb{Z}_2} = a_{i-1} + \delta^i + T^{i+1,2i+s+1}_2$. Corollary 3.2.B implies that

$$T^{i,2i+s-1}_2 - a_{i-1} = T^{i+1,2i+s+1}_2 - a_i \tag{3.4}$$

for all i. Hence, $T^{i,2i+s-1}_2 - a_{i-1} = C$, for some constant C independent of i. Since the support of the Khovanov homology is bounded, there exists i such that $T^{i,2i+s-1}_2 = a_{i-1} = 0$. It follows that C must be zero.

The case of L being a link can be considered similarly. □

3.3. Proof of Corollary 5. Let L be an alternating link with m components. Then its Jones polynomial has the form $J_L(q) = \sum_i c_i q^{2i+\gamma}$, where $\gamma = m \mod 2$

(cf. Corollary 2.2.C). Define $d(L) = |J_L(\sqrt{-1})| = \sum_i |c_i|$. In fact, $d(L) = |\det(L)|$, where $\det(L)$ is the determinant of L, hence the notation.

3.3.A. Theorem (Thistlethwaite [14], Theorem 1). *Let L be a prime non-split alternating link that admits an irreducible alternating diagram with n crossings. Then its Jones polynomial $J_L(q)$ has breadth (that is, the difference between the highest and lowest powers of q) of exactly $2n$ and is alternating. If, moreover, L is not a $(2,k)$-torus link, then $J_L(q)$ has no gaps.*

3.3.B. Lemma. *Let L be an alternating link with m components. Then $d(L) \geq 2^{m-1}$. Moreover, if L is not the trivial knot, the Hopf link, their connected sums or disjoint unions, then $d(L) > 2^{m-1}$.*

Proof. Assume first that L is non-split and prime. It is easy to deduce from (2.2) (see also [5]) that $J_L(1) = \sum_i c_i = (-2)^{m-1}$. Hence, $d(L) \geq |J_L(1)| = 2^{m-1}$. It only remains to show that if L is neither the trivial knot, nor the Hopf link, then the polynomial $J_L(q)$ has both strictly positive and strictly negative coefficients. Theorem 3.3.A implies this for all L except the trivial knot and $(2,k)$-torus links.

Denote a $(2,k)$-torus link by TL_k. It easily follows from (2.2) that $d(TL_k) = k$. Indeed, if one changes a positive crossing of TL_k into a negative one, one gets TL_{k-2} , and if one smoothes such a crossing, one obtains TL_{k-1}. Recall now that a $(2,k)$-torus link has at most 2 components and that the Hopf link is exactly the $(2,2)$-torus link.

Now the Lemma follows from the fact that the Jones polynomial of the connected sum and disjoint union of links L_1 and L_2 is equal to $J_{L_1}(q)J_{L_2}(q)$ and $(q+1/q)J_{L_1}(q)J_{L_2}(q)$, respectively. □

3.3.C. Lemma. *Let L be an alternating link that is not the trivial knot, the Hopf link, their connected sums and disjoint unions. Then $\operatorname{rank}\mathcal{H}(L) > 2^m$, where m is the number of components of L and $\operatorname{rank}\mathcal{H}(L) = \sum_{i,j} h^{i,j}(L)$ is the total rank of the Khovanov homology.*

Proof. Consider first the case when L is non-split. Theorem 2.3.C implies that $\operatorname{rank}\mathcal{H}(L) \geq 2^m$. Assume that this rank is 2^m. In this case $Kh'(L)$ must be 0 and

$$K_L(q) = Kh(L)(-1,q) = q^{-\sigma(L)}\left[(q+1/q)\left(\sum_{E\subset\{2,\ldots,m\}} (-q^2)^{2\sum_{\substack{k\in E\\ l\notin E}} \ell_{k,l}}\right)\right]. \quad (3.5)$$

Since $J_L(q) = K_L(q)/(q+1/q)$, one has

$$d(L) = |J_L(\sqrt{-1})| = \sum_{E\subset\{2,\ldots,m\}} 1^{2\sum_{\substack{k\in E\\ l\notin E}} \ell_{k,l}} = 2^{m-1}. \quad (3.6)$$

This contradicts Lemma 3.3.B. Hence, $\operatorname{rank}\mathcal{H}(L) > 2^m$.

The general case follows from the fact that $\operatorname{rank}\mathcal{H}(L)$ is multiplicative under disjoint union (see [6], Corollary 12). □

Let us now finish the proof of Corollary 5. Lemma 3.3.C states that $\operatorname{rank}\mathcal{H}(L) > 2^m$ and, hence, $Kh'(L) \neq 0$ (in notation of Theorem 2.3.C). Since L is WT-thin by Theorem 3, it follows from 2.3.E that $Kh_T(L) \neq 0$ as well. Hence, L has non-trivial torsion. Since L is WT-thin, some torsion elements must be of order 2. □

4. Torsion of order p of the Khovanov homology

This section is devoted to proving Theorem 1. We start by showing that the Khovanov homology over $\mathbb{Z}_p$ of an H-slim link satisfy Conjecture 1.A as well.

4.1. Khovanov homology with $\mathbb{Z}_p$ coefficients. Let L be an oriented link and p be an odd prime number. Denote by $\mathcal{H}^{i,j}_{\mathbb{Z}_p}(L)$ the Khovanov homology of L over $\mathbb{Z}_p$ and by $h^{i,j}_{\mathbb{Z}_p}(L)$ their Betti numbers. Let $Kh_{\mathbb{Z}_p}(L)(t,q) = \sum_{i,j} t^i q^j h^{i,j}_{\mathbb{Z}_p}(L)$ be the corresponding Poincaré polynomial.

4.1.A. Theorem (cf. Theorem 2.3.C and [9], Theorems 1.1 and 1.2). *Let L be an m-component oriented H-slim link, for example, a non-split alternating link. Then $Kh_{\mathbb{Z}_p}(L)$ satisfies identity (2.7) for the original Khovanov polynomial with some other polynomial $Kh'_p(L)$ instead of $Kh'(L)$. If L is an H-slim knot, then this identity becomes*

$$Kh_{\mathbb{Z}_p}(L) = q^{-\sigma(L)-1}\big(1 + q^2 + (1 + tq^4)Kh'_p(L)(tq^2)\big), \tag{4.1}$$

where $\sigma(L)$ is the signature of L.

Proof. We will show that the methods used by Lee to prove Theorem 2.3.C for the Khovanov homology with $\mathbb{Q}$ coefficients work in our $\mathbb{Z}_p$ case as well if p is odd prime. Only the main steps are to be outlined and the reader is assumed to be familiar with [9].

First of all, we define the Lee differential Φ of bidegree $(1,4)$ on the Khovanov chain complex $\mathcal{C}$. The corresponding incidence numbers $(S_1 : S_2)_\Phi$ of two enhanced states $S_1 \in \mathcal{C}^{i,j}_{\mathbb{Z}_p}(D)$ and $S_2 \in \mathcal{C}^{i+1,j+4}_{\mathbb{Z}_p}(D)$ are defined is a similar way to the original ones from page 7 with the only difference being in condition III:
The incidence number $(S_1 : S_2)_\Phi$ is zero unless all of the following 3 conditions are met, in which case $(S_1 : S_2)_\Phi = \pm 1$ with the sign defined as before:

- I$_\Phi$. the markers from S_1 and S_2 differ at one crossing of D only and at this crossing the marker from S_1 is positive, while the marker from S_2 is negative;
- II$_\Phi$. the common circles of D_{S_1} and D_{S_2} have the same signs;
- III$_\Phi$. one of the following two conditions are met:
  1) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} are positive and the resulting circle from D_{S_2} is negative;
  2) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} is positive and both the resulting circles from D_{S_2} are negative;

It is easy to see that Φ is indeed a differential and (anti)commutes with the Khovanov differential d. Lee's proofs from [9] can be applied to our version of Φ to show that it also commutes with the isomorphisms induced on $\mathcal{H}_{\mathbb{Z}_p}(L)$ by the Reidemeister moves. Hence, Φ gives rise to a well defined differential on $\mathcal{H}_{\mathbb{Z}_p}(L)$.

Consider now yet another differential $\Phi + d$ on $\mathcal{C}_{\mathbb{Z}_p}(D)$. It can be best described by changing the labels on the circles comprising enhanced states from '+' and '−' to $\mathbf{a} = (\text{'+'}) + (\text{'−'})$ and $\mathbf{b} = (\text{'+'}) - (\text{'−'})$. In this notation one has a new third condition on the incidence numbers $(S_1 : S_2)_{\Phi d}$:

- III$_{\Phi d}$. one of the following 4 conditions are met:
  1) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} and the resulting circle from D_{S_2} are marked with $\mathbf{a}$, then $(S_1 : S_2)_{\Phi d} = \pm 2$;

2) $|S_2| = |S_1| - 1$, both joining circles from D_{S_1} and the resulting circle from D_{S_2} are marked with $\mathbf{b}$, then $(S_1 : S_2)_{\Phi d} = \mp 2$;
3) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} and both the resulting circles from D_{S_2} are marked with $\mathbf{a}$, then $(S_1 : S_2)_{\Phi d} = \pm 1$;
4) $|S_2| = |S_1| + 1$, the splitting circle from D_{S_1} and both the resulting circles from D_{S_2} are marked with $\mathbf{b}$, then $(S_1 : S_2)_{\Phi d} = \pm 1$.

Denote by $H(D)$ the homology with respect to $\Phi + d$. It can be shown that $H(D)$ is invariant under the Reidemeister moves, so that we can safely write $H(L)$ instead. The fact that $p \neq 2$ is crucial here, as the proof involves division by 2 (see. [9]).

Theorem 6.1 from [9] that states

$$H(L) \cong \frac{\operatorname{Ker}\bigl(\Phi : \mathcal{H}_{\mathbb{Z}_p}(L) \to \mathcal{H}_{\mathbb{Z}_p}(L)\bigr)}{\operatorname{Im}\bigl(\Phi : \mathcal{H}_{\mathbb{Z}_p}(L) \to \mathcal{H}_{\mathbb{Z}_p}(L)\bigr)} \tag{4.2}$$

still holds true without changes. One needs to use the fact that L is $\mathbb{Z}_p$H-thin, since it is H-slim, here.

The only non-trivial generalization is proving an analogue of Theorem 5.1 of [9] that $\dim_{\mathbb{Z}_p} H(L) = 2^m$. Lee's prove uses Hodge theory arguments that are not applicable to the $\mathbb{Z}_p$ case. Fortunately for us, Hodge theory is only used to provide a *lower* bound on $\dim_{\mathbb{Q}} H(L;\mathbb{Q})$. It follows that $2^m \leq \dim_{\mathbb{Q}} H(L;\mathbb{Q}) \leq \dim_{\mathbb{Z}_p} H(L)$, where the former inequality is provided by Theorem 5.1 from [9], and the latter one by the universal coefficient formula. In particular, all the enhanced states of D, such that at every crossing the two touching circles have different labels, are linearly independent in $H(L)$. Such states are in one-to-one correspondence with all the orientations of L (see [9]). Lee's proof of the fact that $\dim_{\mathbb{Q}} H(L;\mathbb{Q}) \leq 2^m$ still works without changes for $\mathbb{Z}_p$. Hence $2^m \leq \dim_{\mathbb{Z}_p} H(L) \leq 2^m$ and $\dim_{\mathbb{Z}_p} H(L) = 2^m$.

Filling the remaining technical gaps is left to the reader. □

4.2. Proof of Theorem 1. Let L be an H-slim link. Since $h^{i,j}_{\mathbb{Z}_p}(L) = h^{i,j}(L) + T^{i,j}_p(L) + T^{i+1,j}_p(L)$, it follows from Theorems 2.3.C and 4.1.A that $h_{\mathbb{Z}_p}(L) - h(L)$ are arranged in "knight move" pairs everywhere without having to subtract anything (cf., for example, Conjecture 1.A). Hence,

$$T^{i,2i-\sigma(L)-1}_p(L) = T^{i+1,2i-\sigma(L)+3}_p(L) + T^{i+2,2i-\sigma(L)+3}_p(L). \tag{4.3}$$

Since the support of the Khovanov homology is finite, all $T^{i,j}_p(L)$ must be zero. □

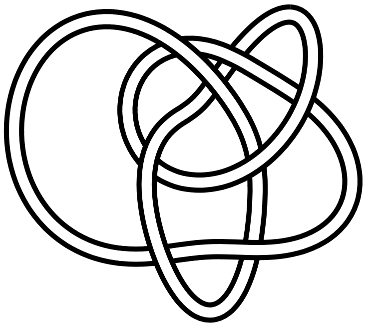
Appendix

This section contains information about standard and reduced Khovanov homology of knots whose torsion has some remarkable properties. The knot pictures below were generated using R. Scharein's program KnotPlot [11].

A.1. How to read the tables. Columns and rows of the tables below are marked with i- and j-grading of the Khovanov homology, respectively. For the standard homology, the j-grading is always odd and the corresponding table entries are printed in boldface. The reduced homology have their j-grading even. They occupy places between the main rows.

Only entries representing non-trivial groups are shown. An entry of the form $a[b,c]$ means that the corresponding group is $\mathbb{Z}^a \oplus \mathbb{Z}_2^b \oplus \mathbb{Z}_4^c$. If some factors are missing from the group, then the corresponding numbers are absent as well.

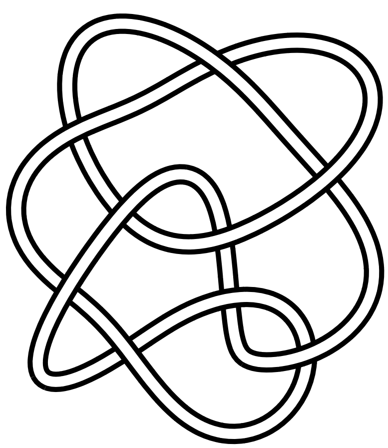
A.2. The knot 8_{19}. This is the first H-thick knot. It is T-thick as well.



	0	1	2	3	4	5
17						**1**
						1
15						**1**
13				**1**	**1**	
				1	1	
11				[**1**]	**1**	
			1			
9			**1**			
7	**1**					
	1					
5	**1**					

Table 1. The knot 8_{19} and its Khovanov homology (standard and reduced).

A.3. The knot 9_{42}. This knot is H-thick but T-thin. Conjecture 3 states that every T-thick knot should be H-thick as well. This example shows that the converse is not true in general.



	-4	-3	-2	-1	0	1	2
7							**1**
							1
5							[**1**]
						1	
3					**1**	**1**	
					1		
1				**1**	**1**[**1**]		
				2	1		
-1				**1**[**1**]	**1**		
			1				
-3		**1**	**1**				
		1					
-5		[**1**]					
	1						
-7	**1**						

Table 2. The knot 9_{42} and its Khovanov homology (standard and reduced).

A.4. The knot $13n_{3663}$. This is the first T-rich knot (the groups with excessive torsion are $\mathcal{H}^{-3,-7}$, $\mathcal{H}^{-3,-5}$, $\mathcal{H}^{-2,-5}$, $\mathcal{H}^{-2,-3}$, $\mathcal{H}^{0,-1}$, $\mathcal{H}^{0,1}$, $\mathcal{H}^{1,1}$, and $\mathcal{H}^{1,3}$). This knot has 2-torsion in the reduced homology as well. This supports the claim of Conjecture 4 that a knot is T-rich iff its reduced Khovanov homology has torsion. This knot is also the first one whose homology are supported on 4 diagonals. The only other knots with 13 crossings or less that share the same properties are $13n_{4587}$, $13n_{4639}$, and $13n_{5016}$.

	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
13														**1**
														1
11														[**1**]
													1	
9												**1**	**1**	
												1		
7										**1**	**1**	[**1**]		
										1	2			
5									**1**	[**1**]	**1**[**1**]			
									2	1				
3								[**1**]	**2**[**1**]	**1**				
								1[1]	1					
1							**2**[**1**]	**1**[**1**]	[**1**]					
							2[1]	1						
-1					**1**	**1**	**1**[**2**]	**1**						
					1	2								
-3					[**2**]	**1**[**1**]								
				1	1[1]									
-5				**1**[**1**]	**1**[**1**]									
				[1]										
-7		**1**		[**1**]										
		1												
-9		[**1**]												
	1													
-11	**1**													

TABLE 3. Standard and reduced Khovanov homology of the knot $13n_{3663}$.

A.5. The $(4,5)$**-torus knot.** This is one of the first knots whose Khovanov homology has torsion of order 4. Its minimal diagram has 15 crossings. There are no knots with 13 crossings or less that have torsion of order other than 2. All the examples of knots with torsion of order 4 that are known so far are braid positive knots. This knot is also T-rich and has 2-torsion in reduced homology (cf. Conjecture 4).

	0	1	2	3	4	5	6	7	8	9	10
29											**[1,0]**
											[1]
27										**1**	**[1,0]**
										1	
25								**1**		**[0,1]**	
								1	1		
23						**1**		**1[1,0]**	**1**		
						1		[1]			
21						**1**	**1**	**[1,0]**			
							1				
19				**1**	**1**		**1**				
				1	1						
17				**[1,0]**	**1**						
			1								
15			**1**								
13	**1**										
	1										
11	**1**										

TABLE 4. Standard and reduced Khovanov homology of the $(4,5)$-torus knot.

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